Journal of Engineering Physics and Thermophysics, Vol. 76, No. 5, 2003

A CONTRIBUTION TO THE COMPUTATION OF THE YOUNG MODULUS

S. O. Gladkov

UDC 520.1+539.4

The shape of a long bending elastically deformed bar of arbitrary cross section and the line of motion of the moving end have been found by solution of the variational problem with one moving boundary with the use of the transversality condition. The results obtained allow alternative recommendations for experimental measurement of the Young modulus of many elastic materials.

Problems on investigation of elastic deformations of bars have been discussed in numerous publications partially reflected in the form of a number of problems in [1]. However, it should be noted that none of the works we know (we do not give a list of publications — there are a large number of them) has raised the question of determining the shape of the line of the moving end for a comparatively thin beam fixed on one side (at point A) which is bending under the action of a certain force applied to the moving end (see Fig. 1). It is the specific shape of the line of displacement of point B that makes it possible to determine the work of the applied force **F** by giving a unique opportunity to calculate the energy of deformation in any (not necessarily small!) displacement, and to find the relationship between the elastic constants (in particular, the Young modulus) and the known geometric and mechanical parameters.

Below we will give the solution of this problem formulated purely mathematically as one problem of the section of variational calculus with a moving boundary.

Let us assume a long bar (see Fig. 1) of length l fixed on one side with the force **F** acting on its one end and causing it to bend. Point B is thus moving and the aim is to determine the extremal $y(x)$ characterizing the shape of the bar, and the trajectory of motion of point B, i.e., the function $\varphi(x)$.

From the function $\varphi(x)$ found we compute the work done by the force **F** as the integral $A = \int [1 + \varphi'^2(x)]^{1/2} dx$ for the set $x \in [l, x_1]$ and set it equal to the energy of elastic deformation $U_{\min}\{y\}$.

As far as the functional $U\{y\}$ itself that describes the potential energy of deformation is concerned, its dependence (according to the general principle of composing an expression for energy invariant relative to the operation of inversion of coordinates) must involve the square of the curvature K.

If the proportionality factor is denoted by α, in accordance with what has been said above we can write the expression

$$U\{y\} = \frac{\alpha}{2} \int_0^{x_1} K^2 \, dx \,, \tag{1}$$

where x_1 is the moving point that is the projection of point B onto the x axis which moves along a certain line $\varphi(x_1)$. The factor 1/2 has been introduced for reasons of convenience.

In the case of a plane curve the curvature is found from the simple expression $K = y''/(1 + y'^2)^{3/2} = 1/R$, where $R(x)$ is the radius of curvature at a certain point with a coordinate (x, y), and the primes on y denote derivatives. From relation (1) it then follows that

$$U\{y\} = \frac{\alpha}{2} \int_0^{x_1} \frac{y''^2}{(1 + y'^2)^3} \, dx \,. \tag{2}$$

Moscow State Regional University, Moscow, Russia, 10a Radio Str., Moscow, 105005, Russia; email: Sglad@newmail.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 76, No. 5, pp. 144–147, September–October, 2003. Original article submitted December 30, 2002.

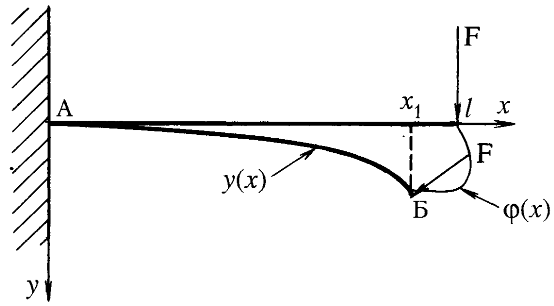

Fig. 1. Diagrammatic representation of the deformation of an elastic bar of prescribed length l in loading by the force **F**.

The given functional enables us to automatically write the Euler–Poisson equation [2] having the form $G_y - \frac{dG_{y'}}{dx} + \frac{d^2 G_{y''}}{dx^2} = 0$, where the function G is equal to $y''^2/(1+y'^2)^3$. It is obvious that the first integral of the equation is $G_{y'} - \frac{dG_{y''}}{dx} = \text{const}$. In expanded form, we have

$$\frac{3y'y''}{(1+y'^2)^4} - \frac{y'''}{(1+y'^2)^3} = \text{const} . \tag{3}$$

Let us select the integration constant equal to zero. This results in the equation $(3y'y'' - y''')(1+y'^2) = 0$. It is easily solved by the substitution of $y' = z(y)$, and for the new function $z(y)$ we have

$$z' = C_1 \frac{(1+z^2)^{3/2}}{z} ,$$

where the prime denotes differentiation with respect to y. It follows that $1 + z^2 = (C_1 y + C_2)^{-2}$. Since $z = y'$, by finding the remaining elementary integral we obtain the sought family of extremal equations

$$(C_1 x + C_3)^2 + (C_1 y + C_2)^2 = 1 . \tag{4}$$

Equation (4), as is obvious, describes the family of circles. In our specific case, if we assume that $y = 0$ for $x = 0$, we easily find the particular solution $x^2 + y^2 - 2R_0 y = 0$. In explicit form, we have

$$y = R_0 + (R_0^2 - x^2)^{1/2} , \tag{5}$$

where the new constant $R_0 = -(C_2/C_1) > 0$ represents the radius of this circle.

Thus, all is clear with the extremal of functional (2); we only note that, according to the sufficient extremum conditions (see [2]), one can state from an analysis of the Weierstrass function: at least a weak minimum of the function $U\{y\}$ (which will be denoted as $U_{\min}$) is attained on the extremals found.

We now find the equation of motion of point B. Since it moves along a certain prescribed line $\varphi(x)$, to determine it we should solve the variational problem with a moving boundary for the functional of the general form

$$V\{y\} = \int_{x_0}^{x_1} G\,(x, y, y', y'')\,dx .$$

A simple analysis [2] leads to the following transversality condition:

$$G + \left(G_{y'} - \frac{dG_{y''}}{dx} \right)(\varphi' - y') + G_{y''}(\varphi'' - y'') \Big|_{x_1} = 0\,, \tag{6}$$

where the notation $G_{y'} = \partial G/\partial y'$ and $G_{y''} = \partial G/\partial y''_y$ has been introduced for the sake of brevity.

Since we are interested in the equation for the function $\varphi(x)$, by solving (6) for this function we obtain

$$\varphi'' - y'' + \left(\frac{G_{y'} - \dfrac{dG_{y''}}{dx}}{G_{y''}} \right)(\varphi' - y') + \frac{G}{G_{y''}} = 0 \tag{7}$$

(we have dropped x_1 here and we consider x to be running).

According to the Euler–Poisson equation, in our case the first integral has been written in the form $G_{y'} - \frac{dG_{y''}}{dx} = C^*$, where the integration constant C^* has been selected (see above) equal to zero. This means that the middle term in (7) disappears. Next, since in the case in question the integrand is $G = y''/(1 + y'^2)^{3/2}$ (here the constant factor $\alpha/2$ is set equal to unity) and the extremal equation is known and prescribed by expression (5), the derivatives are easily found and we obtain

$$G_{y'} = -\frac{6y'y''^2}{(1 + y'^2)^4}\,, \quad G_{y''} = \frac{2y''}{(1 + y'^2)^3}\,, \quad y' = -\frac{x}{(R_0^2 - x^2)^{1/2}}\,, \quad y'' = -\frac{R_0^2}{(R_0^2 - x^2)^{3/2}}\,.$$

Simple computations with the use of Eq. (7) simplified by the absence of the middle term lead us to a simple differential equation of second order for determination of the line of motion of the bar's end: $\varphi'' - (y''/2) = 0$. Its solution will have the form

$$\varphi(x) = \frac{y(x)}{2} + C_1 x + C_2\,. \tag{8}$$

The integration constants are found from the conditions $\varphi(l) = -0$ and $\varphi(x_1) = y(x_1)$. As a result we obtain

$$C_1 = \frac{2R_0 + (R_0^2 - x_1^2)^{1/2} + (R_0^2 - l^2)^{1/2}}{2\,(x_1 - l)}\,, \quad C_2 = -\frac{l\left[R_0 + (R_0^2 - x_1^2)^{1/2}\right] + x_1\left[R_0\,(R_0^2 - l^2)^{1/2}\right]}{2\,(x_1 - l)}\,. \tag{9}$$

If the radius is large (small bending), namely, $R_0 >> x_1$ and l, it approximately follows from (9) that $C_1 \approx 2R_0/(x_1 - l)$ and $C_2 \approx -R_0\,(x_1 + l)/(x_1 - l)$. Furthermore, when $x \in [x_1, l]$, we also have the right to expand the expression for $y(x)$. The latter yields $y(x) \approx 2R_0 - \frac{x^2}{2R_0}$; therefore, confining ourselves just to the term linear in x in the solution (8), we can write that

$$\varphi(x) \approx \frac{2R_0\,(l - x)}{(l - x_1)}\,. \tag{10}$$

To evaluate now the elastic energy stored by the bar and equal (as has been said above) to the work A which is expended by the force **F** on moving point B from the value $x = l$ to $x = x_1$ we should evaluate the simple integral $A = \int F ds$, where the arc element is $dS = (1 + \varphi'^2)^{1/2} dx$. Using (10), we easily obtain $A = F \int \left[1 + \frac{4R_0^2}{(1 - x_1)^2} \right]^{1/2} dx$ for $x \in [x_1, l]$. With allowance for what has been said above we have $A \approx 2R_0^2 F/(l - x_1)$.

On the other hand, the minimum energy stored by the bar as a result of deformation will be equal, according to the determination (2) and the solution (5), to

$$U_{\min} = \frac{\alpha}{2} \int_0^{x_1} \frac{y''^2}{(1 + y'^2)^3}\, dx = \frac{\alpha x_1}{2R_0^2} . \tag{11}$$

From the equation $A = U_{\min}$ we find the coefficient α related to the elastic properties of the bar. One usually assumes that $\alpha = EJ$, where $J = \int \xi^2 d\sigma$ ($d\sigma$ is the element of the cross-sectional area). It then follows from what has been said above that the Young modulus is determined as

$$E = \frac{4FR_0^4}{Jx_1\,(l - x_1)} . \tag{12}$$

We evaluate the expression obtained. We assume that the bar length is l = 100 cm, the moment of inertia of the cross section is J = 50 cm^4, the displacement is x_1 = 10 cm, and the applied force is $F = 10^7$ dyn; let the bending radius be R_0 = 200 cm. Finally, we obtain E = $1.5 \cdot 10^{12}$ erg/cm^3. Such a rough value is very close to the Young modulus for steel.

Finally, we note that if consideration is given to the case where the bending radius R_0 is comparable to the bar length, the consideration presented above needs correction and the work of the force **F** must be written in the form of an integral with allowance for the exact expression for φ′, namely, $\varphi' = C_1 - \dfrac{x}{2\,(R_0^2 - x^2)^{1/2}}$. As a result we have

$$A = \frac{F}{2} \int_{x_1}^{l} \frac{\left[1 - 4C_1 \dfrac{x}{R_0}\left(1 - \dfrac{x^2}{R_0^2}\right)^{1/2} + 4C_1^2\left(1 - \dfrac{x^2}{R_0^2}\right)\right]^{1/2}}{1 - \dfrac{x^2}{R_0^2}}\, dx .$$

Using the substitution $x = R_0 \sin t$, we arrive at the expression

$$A = \frac{FR_0}{2} \int_{t_{\text{low}}}^{t_{\text{up}}} dt \left[1 - 2C_1 \sin 2t + 4C_1^2 \cos^2 t\right]^{1/2} , \tag{13}$$

where $t_{\text{low}} = \arcsin \dfrac{x_1}{R_0}$, $t_{\text{up}} = \arcsin \dfrac{l}{R_0}$, and the constant C_1 has been determined in (9).

We emphasize that formula (13) "works" for any deformations, including very considerable ones, but to determine the Young modulus we should integrate numerically expression (13) in these cases and only after that should we find E from the equality $A = EJx_1/(2R_0^2)$.

We also note that a number of some extremum problems have been described in [3].

We did not introduce a correction for the natural deflection of the bar under gravity for the simple reason that the bar can always be put on a smooth surface and it could be bent in the plane of this surface when gravity is of no importance. The friction force can easily be allowed for using a correction for the force **F**, namely, **F** can be replaced by the difference $\mathbf{F} - \mathbf{F}_{\text{fr}}$, where $\mathbf{F}_{\text{fr}}$ is the friction force equal to the product of k by **N**.

In closing, we can note the following:

(1) using variational principles, we have computed the "trajectory" of the moving point B to which a certain force **F** (causing the bar to bend by a prescribed value x_1 along the x axis) is applied;

(2) we have proposed an alternative method of evaluation of the Young modulus on the basis of formula (12) in which **F**, R_0, x_1, l, and J are known.

NOTATION

A, work done on the elastically deformed bar, J; $C_{1,2,3}$, integration constants; E, Young modulus, N/m^2; **F**, force applied to the bar, N; $\mathbf{F}_{fr}$, friction force, N; G, integrand in the functional; J, moment of inertia of the bar cross section, m^4; K, curvature of the deformed bar, m^{-1}; k, coefficient of friction of the bar against the surface; l, bar length, m; **N**, reaction of the surface, N; R, radius of curvature of the bar, m; R_0, radius of the circle along which the bar is bent, m; S, arc length, m; dS, element of the arc, m; U, energy of the elastically deformed bar, J; U_{min}, minimum energy of deformation, computed for the dependence $y(x)$, J; x, running abscissa of point B, m; x_1, finite abscissa of movement of point B, m; $y(x)$, extremal of the functional, m; α, proportionality factor, J·m; $\varphi(x)$, line of displacement of the bar end, m; ξ, running coordinate of the bar cross section, m; σ, cross-sectional area of the bar, m^2; t, argument of integration; t_{low}, lower limit of integration; t_{up}, upper limit of integration; $y' = dy/dx$, $y'' = d^2y/dx^2$, and $y''' = d^3y/dx^3$, contracted notation of the first, second, and third derivatives; $\partial/\partial y$ and $\partial/\partial y'$, partial derivatives with respect to y and y'; $G_u = \partial G/\partial u$, contracted notation of the partial derivative with respect to the argument u; $V\{y\}$, functional of y (dimensionless). Subscripts: min, minimum; low, lower; up, upper; fr, friction.

REFERENCES

1. L. D. Landau and E. M. Lifshits, *Elasticity Theory* [in Russian], Moscow (1987).
2. L. É. Él'sgol'd, *Differential Equations and Variational Calculus* [in Russian], Moscow (1969).
3. S. O. Gladkov, *Physics of Composites: Thermodynamic and Dissipative Properties* [in Russian], Moscow (1999).